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GENERALIZED HODGE CONJECTURE

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1. Hodge Conjecture

1.1. Let X be a smooth projective variety of dimension n over $\mathbb{C}$. The fact that X is cut out by polynomials in $\mathbb{P}^N$ implies that it contains "many" subvarieties. Let $z^k(X) = z_{n-k}(X)$ be the free group generated by subvarieties of X of dimension $n-k$ (codimension k).

There is a natural map

$$\mathrm{cl}_k : z^k(X) \to H_{2n-2k}(X, \mathbb{Z}) \cong H^{2k}(X, \mathbb{Z}) \tag{1.1}$$

Recall the DeRahm cohomology

$$H^p_{\mathrm{DR}}(X) = \frac{\{\text{closed } p \text{ forms}\}}{\{\text{exact } p \text{ forms}\}} \tag{1.2}$$

Let $i_{\mathbb{Z}}$ be the map $i_{\mathbb{Z}} : H^{2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{C}) \cong H^{2k}_{\mathrm{DR}}(X)$. The composition $i_{\mathbb{Z}} \circ \mathrm{cl}_k$ can be described as follows.

Let $V \subset X$ be a subvariety of codimension k and let $\nu : \widetilde{V} \to V$ be a desingularization of V. Then $i_{\mathbb{Z}}(\mathrm{cl}_k(V))$ defines a linear map $H^{2n-2k}_{\mathrm{DR}}(X) \to \mathbb{C}$:

$$\omega \in H^{2n-2k}_{\mathrm{DR}}(X) \to \int_{\widetilde{V}} \nu^* \omega \tag{1.3}$$

Note that this map is well defined independent of the choice of ν. Indeed,

$$\int_{V \backslash V_{\mathrm{sing}}} \omega = \int_{\widetilde{V}} \nu^* \omega \tag{1.4}$$

Extend this map by linearity and we see that

$$i_{\mathbb{Z}} \circ \mathrm{cl}_k : z^k(X) \to H^{2n-2k}_{\mathrm{DR}}(X)^\vee \cong H^{2k}_{\mathrm{DR}}(X) \tag{1.5}$$

is given by

$$i_{\mathbb{Z}}(\mathrm{cl}_k(\sum \mu_i V_i))(\omega) = \sum \mu_i \int_{\widetilde{V}_i} \nu_i^* \omega \tag{1.6}$$

where $\nu_i : \widetilde{V}_i \to V_i$ is a desingularization of V_i.

The identification of $H^{2n-2k}_{\mathrm{DR}}(X)^\vee$ and $H^{2k}_{\mathrm{DR}}(X)$ follows from

Theorem 1.1 (Poincare Duality)**.** *The pairing*

$$H^p_{\mathrm{DR}}(X) \times H^{2n-p}_{\mathrm{DR}}(X) \to \mathbb{C} \tag{1.7}$$

given by

$$(\omega_1, \omega_2) \to \int_X \omega_1 \wedge \omega_2 \tag{1.8}$$

is nondegenerate.

Let $E_X^k = \{\mathbb{C}^\infty\ k\text{-forms on } X\}$. Since X is a complex manifold, there is a natural decomposition of E_X^k

$$E_X^k = \bigoplus_{p+q=k} E_X^{p,q} \tag{1.9}$$

where $E_X^{p,q}$ are the $\mathbb{C}^\infty$ (p,q)-forms which in terms of local coordinates $(z_1, z_2, ..., z_n)$ are of the form:

$$\sum_{|I|=p,|J|=q} f_{IJ} dz_I \wedge d\bar{z}_J, \qquad \begin{aligned} &I = 1 \le i_1 < i_2 < ... < i_p \le n \\ &J = 1 \le j_1 < j_2 < ... < j_q \le n \\ &dz_I = dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \\ &d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge ... \wedge d\bar{z}_{j_q} \end{aligned} \tag{1.10}$$

The above decomposition descends into cohomology level, i.e.,

Theorem 1.2 (Hodge Decomposition)**.**

$$H_{\mathrm{DR}}^k(X) = \bigoplus_{p+q=k} H^{p,q}(X) \tag{1.11}$$

where

$$H^{p,q}(X) = \frac{\{\omega \in E_X^{p,q} : d\omega = 0\}}{\{d\omega : \omega \in E_X^{p-1,q-1}\}} \tag{1.12}$$

It is obvious that $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Let $\omega \in H^{2n-2k}(X)$. By Hodge Decomposition, $\omega = \sum_{p+q=2n-2k} \omega_{p,q}$ with $\omega_{p,q} \in H^{p,q}(X)$. It is easy to see that $\nu^* \omega_{p,q} = 0$ for all $(p,q) \ne (n-k, n-k)$ in (1.6). Consequently,

$$i_{\mathbb{Z}}(\mathrm{cl}_k(z^k(X))) \subset H^{n-k,n-k}(X)^\vee \cong H^{k,k}(X) \tag{1.13}$$

The isomorphism $H^{n-k,n-k}(X)^\vee \cong H^{k,k}(X)$ follows from Poincare duality by checking the type of ω_1 and ω_2 in (1.8).

Theorem 1.3 (Serre Duality)**.** *The pairing*

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C} \tag{1.14}$$

given by (1.8) *is nondegenerate.*

A more common but slightly confusing way to write (1.13) is

$$\begin{aligned} \mathrm{cl}_k(z^k(X)) \subset H^{k,k}(X, \mathbb{Z}) &= H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X) \\ &= \{\omega \in H^{2k}(X, \mathbb{Z}) : i_{\mathbb{Z}}(\omega) \in H^{k,k}(X)\} \end{aligned} \tag{1.15}$$

A natural question is

Question 1.4. *Is it true*

$$\mathrm{cl}_k(z^k(X)) = H^{k,k}(X, \mathbb{Z})? \tag{1.16}$$

1.2. **Lefschetz** $(1,1)$ **Theorem.** It is known that (1.16) holds for $k=1$. This is so-called Lefschetz $(1,1)$ Theorem. Here is a sketch of the proof.

Since X is smooth, every Weil divisor is Cartier. Hence we have a surjection $z^1(X) \to \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$. By the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{1.17}$$

we have

$$H^1(X,\mathcal{O}^*) \xrightarrow{\alpha} H^2(X,\mathbb{Z}) \xrightarrow{\beta} H^2(X,\mathcal{O}) \cong H^{0,2}(X) \tag{1.18}$$

Obviously, $\beta(H^{1,1}(X,\mathbb{Z})) = 0$ and hence $H^{1,1}(X,\mathbb{Z}) \subset \ker\beta = \mathrm{Im}\,\alpha$. And (1.16) follows.

1.3. **Amendments.** Note that $H^2(X,\mathbb{Z})_{\mathrm{tors}} \subset H^{1,1}(X,\mathbb{Z})$. An example of Grothendieck shows that there is a class $\xi \in H^2(X,\mathbb{Z})_{\mathrm{tors}}$ such that $\xi \notin \mathrm{cl}_k(z^k(X))$. Therefore, (1.16) fails in general. Naturally, one asks whether it holds after we modulo the torsion part:

Question 1.5. *Is the map*

$$z^k(X) \xrightarrow{\mathrm{cl}_k} H^{k,k}(X,\mathbb{Z}) \to H^{k,k}(X,\mathbb{Z})/H^k(X,\mathbb{Z})_{\mathrm{tors}} \tag{1.19}$$

surjective?

However, this is still false.

Theorem 1.6 (J. Kollár and etc)**.** *Let X be a very general threefold in $\mathbb{P}^4$ of degree d. For $d >> 0$, every curve on X has degree divisible by d.*

By weak Lefschetz theorem, $H_2(X,\mathbb{Z}) \cong H^2(\mathbb{P}^4,\mathbb{Z}) = \mathbb{Z}$ is generated by lines $l \subset \mathbb{P}^4$. Let $C \subset X$ be a curve on X. By Kollár's result, $[C] = k[l]$ in $H_2(X,\mathbb{Z})$ for some $d|k$. Therefore, the image of $\mathrm{cl}_k : z^2(X) \to H_2(X,\mathbb{Z}) \cong H^4(X,\mathbb{Z}) = \mathbb{Z}[l]$ is contained in the subgroup generated by $d[l]$. Since $[l]$ is algebraic, $H^{2,2}(X,\mathbb{Z}) = H^4(X,\mathbb{Z})$ is torsion free but cl_k is not surjective.

The amendment of this situation is to consider everything over $\mathbb{Q}$.

Conjecture 1.7 (Hodge Conjecture)**.** *Is the map*

$$\mathrm{cl}_k \otimes \mathbb{Q} : z^k(X)\otimes\mathbb{Q} \to H^{k,k}(X,\mathbb{Q}) = H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X) \tag{1.20}$$

surjective?

If the above conjecture holds for X and k, then we say $\mathrm{GHC}(k, 2k, X)$ holds. We will explain this notation later.

1.4. **Examples.** Next we will give some simple examples where Hodge conjecture is known to hold.

$\mathrm{GHC}(1,2,X)$ holds for any X by Lefschetz $(1,1)$ theorem.

Let E be a vector bundle over X. The cohomology of $\mathbb{P}E$ is generated by $H^*(X)$ and the tautological class μ. Since μ is algebraic, $\mathrm{GHC}(k, 2k, \mathbb{P}E)$ holds if and only if $\mathrm{GHC}(k, 2k, X)$ holds.

Exercise 1.8. $\mathrm{GHC}(k, 2k, X)$ holds for any X with $\dim X \le 3$.

Hint: Use the fact that $H^{1,1}(X,\mathbb{Q}) \times H^{1,1}(X,\mathbb{Q}) \to H^{2,2}(X,\mathbb{Q})$ is surjective for $\dim X = 3$.

Exercise 1.9. Let $f : X \to Y$ be a surjective and generically finite map. If $\mathrm{GHC}(k, 2k, X)$ holds, then $\mathrm{GHC}(k, 2k, Y)$ holds.

Hint: There are two natural maps $f^* : H^k(Y) \to H^k(X)$ and $f_* : H^k(X) \to H^k(Y)$, where f_* is given by $f_* : H_{2n-k}(X) \to H_{2n-k}(Y)$ and duality $H_{2n-k}(X) \cong H^k(X)$ and $H_{2n-k}(Y) \cong H^k(Y)$. Show that $f_* \circ f^* = m$, where m is the degree of f.

Exercise 1.10. Let $\widetilde{X}$ be the blowup of X along a smooth subvariety $Y \subset X$. If $\mathrm{GHC}(k, 2k, X)$ and $\mathrm{GHC}(k, 2k, Y)$ hold, then $\mathrm{GHC}(k, 2k, \widetilde{X})$ holds.

Hint: Let E be the exceptional divisor and $\pi_E : E \to Y$ be the projection. Then we have the exact sequence

$$0 \to H^k(X) \to H^k(\widetilde{X}) \to \pi_E^* H^k(Y) \to 0 \tag{1.21}$$

Exercise 1.11. Let X and Y be two smooth projective fourfolds that are birational to each other. Then $\mathrm{GHC}(2, 4, X)$ holds if and only if $\mathrm{GHC}(2, 4, Y)$ holds.

Exercise 1.12. $\mathrm{GHC}(2, 4, X)$ holds for any uniruled fourfolds X.

A variety X is uniruled if there exists a rational dominant map $Y \times \mathbb{P}^1 \to X$ with $\dim Y = \dim X - 1$.

2. General Hodge Conjecture

2.1. Let $F^l H^k(X) = \oplus_{p \geq l} H^{p,k-p}(X)$. So we have the filtration $F^0 \supset F^1 \supset \ldots \supset F^k \supset \{0\}$.

Exercise 2.1. Check that $F^p \cap \overline{F^q} = H^{p,q}(X)$ for $p + q = k$ and $H^k(X) = F^p \oplus \overline{F^{k-p+1}}$.

It is not hard to see that $H^{k,k}(X,\mathbb{Q}) = F^k H^{2k}(X) \cap H^{2k}(X,\mathbb{Q})$. So we can put Hodge conjecture in the form

$$\mathrm{cl}_k(z^k(X)_{\mathbb{Q}}) = F^k H^{2k}(X) \cap H^{2k}(X,\mathbb{Q}) \tag{2.1}$$

2.2. **Coniveau filtration.** Let $\nu : Y \to X$ be a morphism between two smooth projective varieties X and Y. Then the Gysin map $\nu_* : H^{k-2r}(Y) \to H^k(X)$ is defined by combining $\nu_* : H_{2n-k}(Y) \to H_{2n-k}(X)$ with duality $H_{2n-k}(Y) \cong H^{k-2r}(Y)$ and $H_{2n-k}(X) \cong H^k(X)$, where $\dim X = n$, $\dim Y = m$ and $r = n - m$.

Let Y be a subvariety of X of codimension r. Suppose that Y is smooth. Then we have the exact sequence

$$H^{k-2r}(Y) \to H^k(X) \to H^k(X - Y) \tag{2.2}$$

In case that Y is singular, the above exact sequence still holds if we replace $H^{k-2r}(Y) \to H^k(X)$ by $H^{k-2r}(\widetilde{Y}) \xrightarrow{\nu_*} H^k(X)$, where $\nu : \widetilde{Y} \to Y$ is a desingularization of Y.

Definition 2.2. Let X be a smooth projective variety. The coniveau filtration of $H^k(X,\mathbb{Q})$ is given by

$$\begin{aligned} N^pH^k(X,\mathbb{Q}) &= \sum_{\operatorname{codim}_X Y\geq p} \ker\Big(H^k(X,\mathbb{Q})\to H^k(X-Y,\mathbb{Q})\Big) \\ &= \sum_{\operatorname{codim}_X Y\geq p} \operatorname{Im}\Big(H^{k-2p}(\widetilde{Y},\mathbb{Q})\to H^k(X,\mathbb{Q})\Big) \end{aligned} \tag{2.3}$$

Exercise 2.3. Verify that $N^kH^{2k}(X,\mathbb{Q}) = \operatorname{cl}_k(z^k(X)\otimes\mathbb{Q})$.

Hence Hodge conjecture can be reformulated as

$$N^kH^{2k}(X,\mathbb{Q}) = F^kH^{2k}(X)\cap H^{2k}(X,\mathbb{Q}) \tag{2.4}$$

Exercise 2.4. Show that $N^pH^k(X,\mathbb{Q})\subset F^pH^k(X)$.

Hint: Let Y be a subvariety of X of codimension $r\geq p$ and $\nu:\widetilde{Y}\to Y$ be a desingularization of Y. For any $\omega\in H^{k-2r}(\widetilde{Y})$ and $\varepsilon\in H^{2n-k}(X)$, we have

$$\int_X \nu_*\omega\wedge\varepsilon = \int_{\widetilde{Y}}\omega\wedge\nu^*\varepsilon \tag{2.5}$$

An analysis of the types of ω and ε gives us what we want.

So it is natural to generalize Hodge conjecture in the following way.

Question 2.5. *Is it true that*

$$N^pH^k(X,\mathbb{Q}) = F^pH^k(X)\cap H^k(X,\mathbb{Q}) \tag{2.6}$$

However, this is not true due to a counterexample of Grothendieck.

2.3. **Abstract Hodge structure.**

Definition 2.6. Let $H_\mathbb{R}$ be a finite dimensional vector space over $\mathbb{R}$ with a nondegenerate lattice $H_\mathbb{Z}\subset H_\mathbb{R}$ (or a $\mathbb{Q}$-vector space $H_\mathbb{Q}\subset H_\mathbb{R}$) and let $H_\mathbb{C} = H_\mathbb{R}\otimes\mathbb{C}$. A Hodge structure of weight k is a filtration $H_\mathbb{C} = F^0\subset F^1\subset ...\subset\{0\}$ such that $H_\mathbb{C} = F^l\oplus\overline{F^{k-l+1}}$ for all l.

Exercise 2.7. Let $H^{p,q} = F^p\cap\overline{F^q}$ for $p+q=k$. Show that $H^{p,q} = \overline{H^{q,p}}$ and $H_\mathbb{C} = \oplus_{p+q=k}H^{p,q}$.

Definition 2.8. Let H and L be two Hodge structures of weight h and l, where $l = h+2r$ for some integer r. A morphism of type (r,r) is between H and L is a linear map $\Psi: H_\mathbb{C}\to L_\mathbb{C}$ induced by a map $H_\mathbb{Z}\to L_\mathbb{Z}$ or $H_\mathbb{Q}\to L_\mathbb{Q}$ such that $\Psi(F_H^p)\subset\Psi(F_L^{p+r})$. We call H a sub-Hodge structure of L if $\Psi: H\hookrightarrow L$ is an inclusion.

Let $f: X\to Y$ be a morphism between two smooth projective varieties. Both f^* and f_* induces morphism of hodege structures between $H^\bullet(X)$ and $H^\bullet(Y)$ of proper weights. More generally, every $\xi\in z^k(X\times Y)$ defines a morphism ρ_ξ of Hodge structures of type (r,r) between $H^l(X)$ and

$H^{l+2r}(Y)$, where $\dim X = n$, $\dim Y = m$ and $r = k - n$. Let $[\xi] = \mathrm{cl}_k(\xi) \in H^{2k}(X \times Y, \mathbb{Z})$. By Künneth decomposition, we write

$$[\xi] = \sum_{p+q=2k} [\xi]_{p,q} \tag{2.7}$$

with $[\xi]_{p,q} \in H^p(X,\mathbb{Z}) \otimes H^q(Y,\mathbb{Z})$. By duality, $[\xi]_{p,q}$ induces a map $\rho_\xi : H^{2n-p}(X,\mathbb{Z}) \cong H^p(\mathbb{Z})^\vee \to H^q(Y,\mathbb{Z})$.

Exercise 2.9. Show that ρ_ξ defined as above is a morphism of Hodge structures of type (r,r) between $H^\bullet(X)$ and $H^{\bullet+r}(Y)$. Actually,

$$\rho_\xi(\omega) = (\pi_Y)_*(\pi_X^*\omega \wedge [\xi]) \tag{2.8}$$

for every $\omega \in H^\bullet(X)$, where π_X and π_Y are the projections of $X \times Y$ to X and Y.

Exercise 2.10. Let X be a smooth projective variety and D be divisor on X. Let $\Delta : X \to X \times X$ be the diagonal map and $\xi = \Delta(D)$. Then $\rho_\xi(\omega) = \omega \wedge [D]$.

Exercise 2.11. Let $f : Y \to X$ be a morphism between two smooth projective varieties. Let $G = \{(y, f(y)) : y \in Y\} \subset Y \times X\}$ be the graph of f. Then $\rho_G = f_*$ defines a morphism of Hodge structure of type (p,p) between $H^{\bullet-2p}(Y)$ and $H^\bullet(X)$.

A consequence of this is

Proposition 2.12. *$N^pH^k(X,\mathbb{Q})$ is a sub-Hodge structure of $H^k(X,\mathbb{Q})$.*

It was Grothendieck's observation that the RHS of (2.6) is not a sub-Hodge structure that led to a counterexample and amendment.

Conjecture 2.13 ((Grothendieck Amended) General Hodge Conjecture)**.** GHC(p,k,X)*:*

$$N^pH^k(X,\mathbb{Q}) = F_h^pH^k(X,\mathbb{Q}) \tag{2.9}$$

where $F_h^pH^k(X,\mathbb{Q})$ is the largest sub-Hodge structure of $H^k(X,\mathbb{Q})$ contained in $F^pH^k(X) \cap H^k(X,\mathbb{Q})$.

Obviously, GHC$(p,2p,X)$ is the original Hodge conjecture.

2.4. **Grothendieck's counterexample.** Let $X = E_1 \times E_2 \times E_3$, where $E_i = \mathbb{C}/\Lambda_i$ are complex tori. If $F^1H^3(X) \cap H^3(X,\mathbb{Q})$ is a sub-Hodge structure of $H^3(X,\mathbb{Q})$, $F^1H^3(X) \cap H^3(X,\mathbb{Q})$ has even dimension. Obviously,

$$F^1H^3(X) \cap H^3(X,\mathbb{Q}) \cong \{\gamma \in H_3(X,\mathbb{Q}) : \int_\gamma \omega = 0, \forall \omega \in H^{3,0}(X)\} \tag{2.10}$$

By Künneth, we see that $H_{i_1}(E_1,\mathbb{Q}) \otimes H_{i_2}(E_2,\mathbb{Q}) \otimes H_{i_3}(E_3,\mathbb{Q})$ is contained in the RHS of (2.10) for $i_1 + i_2 + i_3 = 3$ and $(i_1,i_2,i_3) \neq (1,1,1)$. Consequently,

$$\dim_{\mathbb{Q}}(F^1H^3(X) \cap H^3(X,\mathbb{Q})) \equiv \dim_{\mathbb{Q}} V \pmod 2 \tag{2.11}$$

where

$$V = \{\gamma \in H_1(E_1,\mathbb{Q}) \otimes H_1(E_2,\mathbb{Q}) \otimes H_1(E_3,\mathbb{Q}) : \int_\gamma \omega = 0, \forall \omega \in H^{3,0}(X)\} \tag{2.12}$$

Let $\omega = dz_1 \wedge dz_2 \wedge dz_3$, $\Lambda_i =< 1, \tau_i >$ and $\{\alpha_{i0}, \alpha_{i1}\}$ be the generators of $H_i(E_i, \mathbb{Z})$ such that

$$\int_{\alpha_{i0}} dz_i = 1 \text{ and } \int_{\alpha_{i1}} dz_i = \tau_i\tau_i \tag{2.13}$$

Let $\beta_{j_1j_2j_3} = \alpha_{1j_1} \times \alpha_{2j_2} \times \alpha_{3j_3}$ be the eight generators of

$$H_1(E_1,\mathbb{Z}) \otimes H_1(E_2,\mathbb{Z}) \otimes H_1(E_3,\mathbb{Z}) \cong \mathbb{Z}^8 \tag{2.14}$$

Then

$$\int_{\beta_{j_1j_2j_3}} \omega = \tau_1^{j_1}\tau_2^{j_2}\tau_3^{j_3} \tag{2.15}$$

Let $\gamma = \sum r_{j_1j_2j_3}\beta_{j_1j_2j_3}$ with $r_{j_1j_2j_3} \in \mathbb{Q}$. Then $\int_\gamma \omega = 0$ imposes the condition

$$\sum r_{j_1j_2j_3}\tau_1^{j_1}\tau_2^{j_2}\tau_3^{j_3} = 0 \tag{2.16}$$

If we let $\tau_1 = \tau_2 = \tau_3 = \tau$, then

$$r_{000} + (r_{001} + r_{010} + r_{100})\tau + (r_{011} + r_{101} + r_{110})\tau^2 + r_{111}\tau^3 = 0 \tag{2.17}$$

Choose τ to be an algebraic number of degree 3, say $\tau^3 = 2$. Then

$$\begin{aligned} r_{000} + 2r_{111} &= 0 \\ r_{001} + r_{010} + r_{100} &= 0 \\ r_{011} + r_{101} + r_{110} &= 0 \end{aligned} \tag{2.18}$$

Consequently, $\dim_{\mathbb{Q}} V = 8 - 3 = 5$ is not even. Contradiction.

3. GHC for Hypersurfaces and Cylinder map

3.1. Here we study a special case of general Hodge conjecture. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. By weak Lefschetz theorem, the Hodge structures of $H^k(X)$ and $H^k(\mathbb{P}^n)$ are isomorphic for $k < n$. The only interesting cohomology is the middle one:

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X) \tag{3.1}$$

A nontrivial computation shows that $H^{p,q}(X) = 0$ for $\min(p,q) < k = \lfloor (n+1)/d \rfloor$. Consider GHC$(l,n,X)$. The smallest l such that GHC(l,n,X) makes sense is $l = k$.

Question 3.1. *Does* GHC(k, n, X) *hold? That is, is it true*

$$N^k(H^n(X,\mathbb{Q})) = F^k H^n(X) \cap H^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})? \tag{3.2}$$

Or equivalently, does there exist a smooth projective variety Y and a map $\nu : Y \to X$ of codimension k such that

$$\nu_* : H^{n-2k}(Y,\mathbb{Q}) \to H^n(X,\mathbb{Q}) \tag{3.3}$$

is surjective?

GHC(k, n, X) is yet unknown for all n and d. Only some special cases are known. One approach to GHC(k, n, X) is using Fano varieties of k-planes on X. This yields the following theorem.

Theorem 3.2 (Lewis). *If*

$$(k+1)(n+1-k) - \binom{d+k}{k} \ge n - 2k \tag{3.4}$$

the GHC(k, n, X) *holds, where $k = \lfloor (n+1)/d \rfloor$. More precisely, let $\Omega_X(k) \subset \mathbb{G}(k, n+1)$ be the Fano variety parameterizing k-planes contained in X. Then the cylinder map*

$$\Phi : H_{n-2k}(\Omega_X(k),\mathbb{Q}) \to H_n(X,\mathbb{Q}) \tag{3.5}$$

is surjective, where Φ is defined by

$$\Phi(\gamma) = \cup_{[\Lambda]\in\gamma} \Lambda \tag{3.6}$$

To see why (3.5) implies (3.3), let $S = \{(p, \Lambda) : \Lambda \in \Omega_X(k), p \in \Lambda\}$ be the universal family over $\Omega_X(k)$. By weak Lefschetz theorem, there is a subvariety $W \subset \Omega_X(k)$ of dimension $n - 2k$ such that $H_{n-2k}(W,\mathbb{Q}) \twoheadrightarrow H_{n-2k}(\Omega_X(k),\mathbb{Q})$. Let $Y = \pi^{-1}(W)$ where π is the projection $S \to \Omega_X(k)$. Then we have the commutative diagram (over $\mathbb{Q}$)

$$\begin{array}{ccccc} H_{n-2k}(W) & \twoheadrightarrow & H_{n-2k}(\Omega_X(k)) & \xrightarrow{\Phi} & H_n(X) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & \nearrow & \\ H_n(Y) & \longrightarrow & H_n(S) & & \end{array} \tag{3.7}$$

Therefore, $H^{n-2k}(Y) \cong H_n(Y) \to H_n(X) \cong H^n(X)$ is a surjection.

Exercise 3.3. Let X be a general hypersurface in $\mathbb{P}^{n+1}$ of degree d. Then $\Omega_k(X)$ is smooth of pure dimension

$$\delta = (k+1)(n+1-k) - \binom{d+k}{k} \tag{3.8}$$

If $\delta > 0$ and $d > 2$, $\Omega_k(X)$ is connected and hence irreducible.

3.2. Here is a sketch of the proof of Lewis' theorem.

Let $Z \subset \mathbb{P}^{n+2}$ be a general hypersurface of degree d and $X = Z \cap \Gamma$ be a general hyperplane section of Z. The basic idea is to construct two smooth projective varieties $\widetilde{X}$ and $P(Z)$ with diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\gamma} & P(Z) \\ \downarrow \pi & & \downarrow \pi_Z \\ X & \xrightarrow{j} & Z \end{array} \tag{3.9}$$

where $\widetilde{X}$ and $P(Z)$ dominate X and Z via generically finite maps. By weak Lefschetz, $H_n(Z, \mathbb{Z}) \cong \mathbb{Z}$. And it is easy to see that $N^k(H_n(X, \mathbb{Z})) \to H_n(Z, \mathbb{Z})$ is surjective. Therefore,

$$H_n(X, \mathbb{Q}) = N^k(H_n(X, \mathbb{Q})) \Leftrightarrow \ker j_* \subset N^k(H_n(X, \mathbb{Q})) \tag{3.10}$$

Exercise 3.4. Show that

$$\ker \gamma_* \subset N^k(H_n(\widetilde{X}, \mathbb{Q})) \Rightarrow \ker j_* \subset N^k(H_n(X, \mathbb{Q})) \tag{3.11}$$

So it suffices to construct $\widetilde{X}$ and $P(Z)$ with the property

$$\ker \gamma_* \subset N^k(H_n(\widetilde{X}, \mathbb{Q})) \tag{3.12}$$

Here is the construction.

Let $\Omega_Z(k)$ be the Fano variety of k-planes contained in Z. It is easy to check from (3.4) that Z is covered by k-planes.

Let Ω_Z be the subvariety of $\Omega_Z(k)$ of dimension $n + 1 - k$ cut out by generic ample divisors.

Let $P(Z) = \{(p, \Lambda) : p \in \Lambda, \Lambda \in \Omega_Z\}$. Then $\dim P(Z) = n + 1$ and $P(Z)$ dominates Z.

Let $\widetilde{X} = \pi_Z^{-1}(X)$, where π_Z is the map $P(Z) \to Z$. It is easy to see that $\dim \widetilde{X} = n$ and $\widetilde{X}$ dominates X.

Let $\Omega_X = \Omega_X(k) \cap \Omega_Z$ and $P(X) = \rho_Z^{-1}(\Omega_X)$. It is easy to check that $\dim \Omega_X = n - 2k$ and $\dim P(X) = n - k$.

Exercise 3.5. $P(Z)$ is a $\mathbb{P}^k$ bundle over Ω_Z and $\widetilde{X} - P(X)$ is a $\mathbb{P}^{k-1}$ bundle over $\Omega_Z - \Omega_X$.

Consequently, we have the injection

$$H_n(\widetilde{X} - P(X)) \hookrightarrow H_n(P(Z) - P(X)) \tag{3.13}$$

Then (3.12) follows from the diagram

$$\begin{array}{ccccc} H_n(P(X)) & \longrightarrow & H_n(\widetilde{X}) & \longrightarrow & H_n(\widetilde{X} - P(X)) \\ \| & & \downarrow & & \downarrow \\ H_n(P(X)) & \longrightarrow & H_n(P(Z)) & \longrightarrow & H_n(P(Z) - P(X)) \end{array} \tag{3.14}$$

Here we use Borel-Moore homology.

3.3. **Quintic 9-folds.** The first pair (n, d) for which (3.4) fails is $(9, 5)$. As above, let $Z \subset \mathbb{P}^{11}$ be a general quintic 10-fold and let $X = Z \cap \Gamma$ be a general hyperplane section of Z.

The above arguement does not go through due to the fact Z is not covered by 2-planes. Instead of 2-planes, we use rational surfaces to cover Z. At the moment, we use quintic surfaces with a singularity of order 4. Tentatively, we have

Theorem 3.6 (Chen, Lewis)**.** GHC$(2, 9, X)$ *holds for a smooth quintic 9-fold* $X \subset \mathbb{P}^{10}$.

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